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Neutral Multi-Instanton as a Bridge from Weak to Strong Coupling phase in Two Dimensional QCD

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Abstract

Using a contour integral representation we analyze the multi-instanton sector in two dimensional $U(N)$ Yang-Mills theory on a sphere and argue the role of multi-instanton in the large N phase transition. In the strong coupling region at the large N , we encounter “singular saddle point”. Because of this situation, “neutral” configurations of the multi-instanton are dominant in this region. Based on the “neutral” multi-instanton approximation we numerically calculate the multi-instanton amplitude , the free energies and the Wilson loops for finite N . We also compare our results with the large N exact solution of the free energy and the Wilson loop and argue some problems. We find the “neutral” multi-instanton contribution bridges the gap between weak and strong coupling phase.

1 Introduction

In the past years, various attempts to solve low energy QCD have done. Among them key idea is a kind of string description of QCD because the string description automatically implies the confinement. Toward this end , there was remarkable progress for QCD in two dimension in the last few years. In particular Gross and Taylor discovered the string description of two dimensional $SU(N)$ or $U(N)$ Yang-Mills theory in terms of $1/N$ expansion[1]. It is well known that the partition function (and the Wilson loop) of the Yang-Mills theory on genus G Riemann surface Σ_G can be expressed by the form of the sum over irreducible representations of the gauge group[2]. The form is called the heat kernel representation . They showed that expanding the heat kernel by $1/N$, one gets the sum over the branched covering maps of the Riemann surface Σ_G weighted by $N^{2-2g} \exp(-\frac{\lambda}{2} A_{\text{cover}})$. Here g and A_{cover} are genus and area of the covering space respectively. That is, a no-fold string theory with target space Σ_G . But later it was discovered by Douglas and Kazakov that on a sphere this system undergoes a 3rd order phase transition in the large N limit[3]. They showed whereas in the strong coupling phase the large N solution of the free energy has the string expansion corresponded to the map from sphere to sphere , in the weak coupling phase the large N solution is trivial and has no such expansion. Thus it is impossible to interpret QCD as the string theory in the weak coupling phase.

Viewed from the strong coupling phase ,the string sum becomes divergent at the phase transition point due to the entropy of the branch points[4]. So the entropy of the branch points physically causes the phase transition. On the other hand ,viewed from the weak coupling phase instanton induces the phase transition[5, 6]. We recall Witten's work on two dimensional QCD[7]. He showed that the Yang-Mills partition function on Riemann surface can be expressed by a sum over the instanton. The weight associated with the instanton is correctly determined from the heat kernel representation[5]. Using this formula Gross-Matysin showed the following results[6]. The 0 instanton sector gives the weak coupling result for the free energy. In the weak coupling region the 1 instanton amplitude of charge ± 1 is strongly suppressed in the large N limit but at the phase transition point the damping factor disappears .

In addition one can see using their results whereas the damping factor is dis-

peared at the phase transition point, the 1 instanton amplitude of charge ± 1 still has order $N^{-1/2}$ in the strong coupling region. And it's contribution to the free energy is order $N^{-5/2}$ and thus negligible in the large N limit. Hence the 1-instanton effect is insufficient to explain the large N phase transition and it is important to investigate the effect of the multi-instanton in the large N limit. That is our motivation for analyzing the multi-instanton sector.

Since in the strong coupling phase Yang -Mills theory has the string description, by studying the phase transition we can see how the nonperturbative effect of the multi-instanton construct the string picture. That is instructive for real four dimensional QCD. In four dimensional QCD it is unlikely that there is such phase transition[6]. But perturbation can not lead to the string picture, it is important to know how the nonperturbative effect construct the string picture. By considering the case of two dimension in detail, there might give some insight for four dimensional QCD. We should keep the above thing in mind when we study the large N phase transition in two dimensional QCD.

In two dimension the Yang-Mills theory has no transverse gluon and it is in some sense topological. So coupling to various matter fields is very important. But such systems are obtained by considering random walk for the Wilson loop of the Yang-Mills theory[8], our method might become another approach to the above systems.

In the preceeding paper the author gave a scenario of the large N phase transition which explains the $O(1)$ contribution to the free energy in the strong coupling region using the multi-instanton[9]. In this paper following the preceeding paper, we continue our analysis on the multi-instanton contribution to the free energy and the Wilson loop and find out that there is a sort of neutrality in the strong coupling phase at the large N . That is, “neutral” configurations of the multi-instanton are dominated for both the free energy and the Wilson loop in this region. Based on the “neutral” multi-instanton scheme, we also calculate the free energy and the Wilson loop for gauge group $U(3)$, $U(4)$ and $U(5)$. And we compare our results with the large N exact solution of the free energy[3] and the Wilson loop[10] and find that the neutral multi-instanton bridges the gap between the weak and the strong coupling phase in two dimensional

QCD on sphere.

The content of this paper is as follows. In section 2 the formula of the multi-instanton amplitude and the partition function are obtained in terms of contour integral. In section 3 the formula of the Wilson loop average are also obtained in terms of contour integral. In section 4 nature of the 1-instanton and the dilute gas approximation are considered. In section 5 the structure of the multi-instanton amplitude in the large N limit is analyzed in detail. In section 6 numerical calculation of the multi-instanton amplitude , the free energy and the Wilson loop are performed. Also our results are comared to the large N exact solution . In conclusion we sumary the results.

2 partition function

The partition function of two dimensional $U(N)$ Yang-Mills theory has so called the heat kernel representation. The representation was first considered by A.Migdal on disk in the context of the real space renormalization group and later generalized to arbitraly 2-manifold by B.Rusakov[2]. On a sphere with area A , the partition function becomes

$$\begin{aligned} Z(A) &= \int \mathcal{D}A_\mu \exp\left(-\frac{N}{4\lambda} \int d^2x \sqrt{g} \text{tr} F_{\mu\nu} F^{\mu\nu}\right) \\ &= \sum_R (\dim R)^2 \exp\left(-\frac{\lambda A}{2N} C_2(R)\right). \end{aligned} \quad (1)$$

Here R is irreducible representation of the gauge group $U(N)$, $\dim R$ is the dimension of representation R and $C_2(R)$ is the value of the second Casimir operator of rep R . In terms of the highest weight components of R , $n_1 \geq n_2 \geq \dots \geq n_N$, $\dim R$ and $C_2(R)$ are given by

$$\dim R = \prod_{1 \leq i \leq j \leq N} \left(1 - \frac{n_i - n_j}{i - j}\right), \quad (2)$$

$$C_2(R) = \sum_{i=1}^N n_i(n_i + N + 1 - 2i). \quad (3)$$

If we define l_i as $n_i - i + \frac{N+1}{2}$, the partition function becomes ¹

$$Z(A) = \text{const} \cdot e^{\frac{A}{24}(N^2-1)} \sum_{l_1 > \dots > l_N} \prod_{1 \leq i \leq j \leq N} \Delta^2(l) \exp\left(-\frac{A}{2N} \sum_{i=1}^N l_i^2\right). \quad (4)$$

We remark that the condition that l_i 's satisfy the relation $l_1 > l_2 > \dots > l_N$ is irrelevant because the configuration such as $l_i = l_j$ does not contribute to the Van der Monde determinant Δ and $Z(A)$ has the index permutation symmetry. Then the sum becomes free sum over \mathbf{Z}^N for odd N or sum over $(\mathbf{Z} + \frac{1}{2})^N$ for even N and we can rewrite it by the Poisson resumation formula. This is a sort of duality transformation. We get the following expression ² [5]:

$$Z(A) = e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} \sum_{\{m_i\} \in \mathbf{Z}^N} \epsilon^{\sum_{i=1}^N m_i} e^{-\frac{2\pi^2 N}{A} \sum_{i=1}^N m_i^2} w(\{m\}), \quad (5)$$

$$\begin{aligned} w(\{m\}) &\equiv \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \prod_{i < j}^N (y_{ij}^2 - 4\pi^2 m_{ij}^2) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m_i) \Delta(y_i - 2\pi m_i) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \end{aligned} \quad (6)$$

where $y_{ij} = y_i - y_j$, $m_{ij} = m_i - m_j$, $\epsilon = 1$ for odd N and $\epsilon = -1$ for even N . In [5, 6, 7] it was showed that $\{m\}$ corresponde to all Euclidean classical solutions (which we call instanton) up to gauge transformation. So the instanton is dual to the highest weight component. The instanton have nonperturbative effect with respect to both $1/N$ and λ . Hereafter we call m_i instanton charge and call number of nonzero m_i 's instanton number. In the following we analyze the phase transition using the duality and see the instanton induced the phase transition. That is reminiscent of the duality transformation of classical XY model and the Kosterliz-Thouless phase transition [12]. In this case viewed from the low temperature the vortex condensation causes the phase transition.

The multi-instanton amplitude $w(\{m\})$ looks like the partition function of the Gaussian Hermite matrix model but there is a deformation in the Van der Monde determinant Δ . In this section we rewrite $w(\{m\})$ using the method of ortho-polynomial (in

¹Hereafter we absorb λ into A .

²A similar expression was obtained by M. Caselle et al [11]. They showed that the phase transition is due to the winding modes of “fermion on circle”.

this case Hermite polynomial) and obtain a new contour integral representation. The new representation makes clear the role of the multi-instanton in the large N phase transition.

First using the property of the Van der Monde determinant we obtain :

$$w(\{m\}) = \sum_{\mu \in S_N} \operatorname{sgn} \mu \sum_{\sigma \in S_N} \prod_{i=1}^N \int_{-\infty}^{\infty} dy_i P_{\sigma(i)}(y_i + 2\pi m_i) P_{\mu \circ \sigma(i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2}, \quad (7)$$

where S_N is the permutation group on N object and

$$P_n(x) = \frac{1}{2^n \left(\frac{N}{2A}\right)^{\frac{n}{2}}} H_n\left(\sqrt{\frac{N}{2A}}x\right) = x^n + (\text{lower power}) \quad (8)$$

is the ortho-polynomial under the Gaussian weight ;

$$\int_{-\infty}^{\infty} dy P_n(y) P_m(y) e^{-\frac{N}{2A} y^2} = h_n \delta_{nm} = \sqrt{2\pi} \left(\frac{A}{N}\right)^{n+\frac{1}{2}} n! \delta_{nm}. \quad (9)$$

In the following let us assume number of non-zero m_i 's is k i.e. number of instanton is k . Using the permutation symmetry nonzero m_i 's are driven to m_i 's from $i = 1$ to $i = k$. Then we get,

$$\begin{aligned} & w(m_1, \dots, m_k, 0, \dots, 0) \\ &= \sum_{\mu \in S_N} \operatorname{sgn} \mu \sum_{\sigma \in S_N} \prod_{i=k+1}^N h_{\sigma(i)} \delta_{\sigma(i), \mu \circ \sigma(i)} \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{\sigma(i)}(y_i + 2\pi m_i) P_{\mu \circ \sigma(i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2} \\ &= (N-k)! \sum_{\mu \in S_k} \operatorname{sgn} \mu \sum_{a_1 \neq \dots \neq a_k} h_0 \cdots \check{h}_{a_1} \cdots \check{h}_{a_k} \cdots h_{N-1} \\ & \quad \times \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{a_i}(y_i + 2\pi m_i) P_{\mu(a_i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2}. \end{aligned} \quad (10)$$

Here $\mu \in S_k$ is the element of the permutation group acting on the set $\{a_1, \dots, a_k\}$. Using the Taylor series expansion of the Hermite polynomial, we obtain for the above integral,

$$\begin{aligned} & \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{a_i}(y_i + 2\pi m_i) P_{\mu(a_i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2} \times (h_{a_1} \cdots h_{a_k})^{-1} \\ &= \prod_{l=1}^k \sum_{i_l=0}^{\min(a_l, \mu(a_l))} \frac{\mu(a_l) C_{i_l}}{(a_l - i_l)!} \left(\frac{2\pi m N}{A}\right)^{a_l - i_l} (-2\pi m)^{\mu(a_l) - i_l}. \end{aligned} \quad (11)$$

The each series in the above equation can be represented as a contour integral by the following transformation formula.

$$\sum_{i=0}^{\min(a,b)} \frac{{}_b C_i}{(a-i)!} (\alpha)^{a-i} (\beta)^{b-i} = \oint \frac{dt}{2\pi i} e^{\alpha\beta t} \frac{1}{t} \left(\frac{1}{\beta t}\right)^a (\beta(t+1))^b, \quad (12)$$

where the contour of t encircles only the origin counterclockwise. Using this formula, we obtain the following contour integral representation for the k -instanton amplitude.

$$\begin{aligned} & w(m_1, \dots, m_k, 0, \dots, 0) \\ = & \frac{(N-k)!}{N!} Z_G\left(\frac{N}{A}\right) \sum_{a_1 \neq \dots \neq a_k}^{N-1} \sum_{\mu \in S_k} \operatorname{sgn} \mu \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_k}{2\pi i} \frac{1}{t_1 \dots t_k} \\ & \times e^{-\frac{4\pi^2 N}{A} (m_1^2 t_1 + \dots + m_k^2 t_k)} \left(\frac{m_{\mu(1)}}{m_1} \frac{1+t_{\mu(1)}}{t_1}\right)^{a_1} \dots \left(\frac{m_{\mu(k)}}{m_k} \frac{1+t_{\mu(k)}}{t_k}\right)^{a_k}. \end{aligned} \quad (13)$$

Here $Z_G(\beta) \equiv \int \prod dx \Delta^2(x) \exp(-\frac{\beta}{2} \sum x^2)$ is the partition function of the Gaussian Hermite matrix model. We remark the configurations such as $a_1 = a_2$ do not affect the above equation. Hence we can replace $\sum_{a_1 \neq \dots \neq a_k}^{N-1}$ to independent sum $\sum_{a_1}^{N-1} \dots \sum_{a_k}^{N-1}$. In that expression there are many pole free terms. After eliminating them, the amplitude becomes

$$\begin{aligned} = & \frac{(N-k)!}{N!} Z_G\left(\frac{N}{A}\right) \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_k}{2\pi i} e^{-\frac{4\pi^2 N}{A} (m_1^2 t_1 + \dots + m_k^2 t_k)} \left(1 + \frac{1}{t_1}\right)^N \dots \left(1 + \frac{1}{t_k}\right)^N \\ & \times \sum_{\mu \in S_k} \operatorname{sgn} \mu \frac{1}{\frac{m_{\mu(1)}}{m_1} (1+t_{\mu(1)}) - t_1} \dots \frac{1}{\frac{m_{\mu(k)}}{m_k} (1+t_{\mu(k)}) - t_k}. \end{aligned} \quad (14)$$

The last sum is just determinant of matrix M which has (ij) element

$$M_{ij} \equiv \frac{1}{\frac{m_j}{m_i} (1+t_j) - t_i}. \quad (15)$$

It's diagonal element is 1 and this determinant has all information about the interaction between instantons.

For the sake of numerical calculation we further rewrite the k -instanton amplitude in terms of another contour integral. For this purpose, we classify elements of the permutation group S_k into the conjugacy classes. By symmetry argument, the contribution from different elements belonging to same conjugacy class is same. Then we

obtain,

$$\begin{aligned}
& \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \dots + m_k} e^{-\frac{2\pi^2 N}{A} (m_1^2 + \dots + m_k^2)} w(m_1, \dots, m_k, 0, \dots, 0) \\
&= \frac{(N-k)!}{N!} Z_G\left(\frac{N}{A}\right) \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \dots + m_k} \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_k}{2\pi i} e^{-N(\tilde{\Phi}_{m_1}(t_1) + \dots + \tilde{\Phi}_{m_k}(t_k))} \\
&\quad \times \sum_{\text{conj.class}} \text{sgn}[\sigma] T[\sigma] M_{1\sigma(1)} M_{2\sigma(2)} \dots M_{k\sigma(k)},
\end{aligned} \tag{16}$$

where

$$\tilde{\Phi}_m(t) \equiv \Phi_m(t) + \frac{2\pi^2 m^2}{A} \equiv \frac{4\pi^2 m^2}{A} \left(t + \frac{1}{2}\right) - \log\left(1 + \frac{1}{t}\right). \tag{17}$$

We assume σ has cycle structure $[1^{\sigma_1} \dots k^{\sigma_k}] (\sigma_1 + \dots + k\sigma_k = k)$ and put $T[\sigma]$ as number of elements in the conjugacy class which σ belongs to,

$$T[\sigma] = \frac{k!}{1^{\sigma_1} \sigma_1! \dots k^{\sigma_k} \sigma_k!}, \tag{18}$$

$$\text{sgn}\sigma = (-)^{\sigma_2 + \sigma_4 + \dots + \sigma_{2[\frac{k}{2}]}} \tag{19}$$

$$\sum_{\text{conjclass}} = \sum_{\substack{\sigma_1, \dots, \sigma_k = 0 \\ \sigma_1 + \dots + k\sigma_k = k}}. \tag{20}$$

We use (12345)(67) type element as the representative element of the conjugacy class. Because the multiple integral factorize according to the cycle structure of σ , the multi-instanton amplitude is exponentiated with a constraint ($\sigma_1 + \dots + k\sigma_k = k$) which is expressed by a contour integral of z . We get the following result:

$$\begin{aligned}
& \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \dots + m_k} e^{-\frac{2\pi^2 N}{A} (m_1^2 + \dots + m_k^2)} w(m_1, \dots, m_k, 0, \dots, 0) \\
&= (NC_k)^{-1} Z_G\left(\frac{N}{A}\right) \oint \frac{dz}{2\pi i} \frac{1}{z^{k+1}} \exp\left(\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j(A)\right),
\end{aligned} \tag{21}$$

where the contour of z encircles only the origin counterclockwise and

$$\alpha_j(A) \equiv \sum_{m_1, \dots, m_j \neq 0} \epsilon^{m_1 + \dots + m_j} \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_j}{2\pi i} e^{-N(\tilde{\Phi}_{m_1}(t_1) + \dots + \tilde{\Phi}_{m_j}(t_j))} M_{12} M_{23} \dots M_{j1} \tag{22}$$

is the “connected” amplitude of the j instantons. In the eq(21) we extrapolate the range of summation inside the exponential from $\sum_{j=1}^k$ to $\sum_{j=1}^N$, but this does not affect the equation due to the z integral.

Using the above equation we finally obtain for the partition function

$$\begin{aligned}
Z(A) &= e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} \sum_{k=0}^N {}_N C_k \sum_{m_1, \dots, m_k \neq 0} e^{\sum_{i=1}^k m_i} e^{-\frac{2\pi^2 N}{A} \sum_{i=1}^k m_i^2} w(m_1, \dots, m_k, 0, \dots, 0) \\
&= Z_{\text{weak}}(A) \oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1-z} \exp\left(\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j(A)\right),
\end{aligned} \tag{23}$$

where

$$Z_{\text{weak}}(A) \equiv e^{\frac{A}{24}(N^2-1)} Z_G(A/N) = \text{const } e^{N^2(\frac{A}{24} - \frac{1}{2} \log A) - \frac{A}{24}}. \tag{24}$$

Here we use the the dual property of the Hermite Gaussian matrix model as $Z_G(\beta) = \beta^{-N^2} Z_G(1/\beta)$. Since $Z_{\text{weak}}(A)$ is the 0-instanton (i.e. all $m_i = 0$) contribution to the partition function, $Z_{\text{weak}}(A)$ is just the equation (4) before the duality transformation in which the discrete sum is replaced by the continuous integral. That is the peculiar property of the Poisson resumation formula. Douglas-Kazakov showed that $Z_{\text{weak}}(A)$ is in fact the partition function in the weak coupling phase at the large N . Thus the multi-instanton effect is essential in the strong coupling region at the large N .

3 Wilson loop

In this section, following the preceding method, we give the contour integral representation of the multi-instanton sector in the Wilson loop. The heat kernel representation gives the following expression for the Wilson loop on a sphere [2] :

$$\begin{aligned}
W_n(A_1, A_2) &\equiv \left\langle \frac{1}{N} \text{tr}(P e^{i \oint A})^n \right\rangle \\
&= \frac{1}{Z(A)} \sum_{R,S} \text{dim } R \text{dim } S e^{-\frac{A_1 C_2(R) + A_2 C_2(S)}{2N}} \frac{1}{N} \int dU \chi_R(U) \chi_S(U^\dagger) \text{tr } U^n.
\end{aligned} \tag{25}$$

Here the loop devide the sphere with area A into two disks with area A_1, A_2 . The above holonomy integral gives the linear combination of the Clebsh-Gordan coefficient. For example of $n = 1$ case, The sum over representation R, S becomes that over R multiplied by the sum over all possible attachment of one box to the Young tableau of R . Hence in the same way as the partition function, the sum becomes free sum over \mathbf{Z}^N for odd N or $(\mathbf{Z} + \frac{1}{2})^N$ for even N . By the Poisson duality we get the following

result of the Wilson loop average [13].

$$\begin{aligned}
& W_n(A_1, A_2)Z(A) \\
&= e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} \frac{1}{N} \sum_{k=1}^N \sum_{\{m\} \in \mathbf{Z}^N} e^{\sum_{i=1}^N m_i} e^{-\frac{2\pi^2 N}{A} \sum_{i=1}^N m_i^2 - 2\pi i n m_k \frac{A_2}{A} - \frac{n^2 A_1 A_2}{2NA}} \\
&\quad \times \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m_i + \frac{inA_2}{N} \delta_{ik}) \Delta(y_i - 2\pi m_i + \frac{inA_1}{N} \delta_{ik}) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \quad (26)
\end{aligned}$$

where $\frac{1}{N} \sum_{k=1}^N \exp(-2\pi i n m_k \frac{A_2}{A})$ is the classical value of the Wilson loop with the multi-instanton configuration. From the above equation, the Wilson loop average apparently has $A_1 \leftrightarrow A_2$ exchange symmetry. We also find that calculating the Wilson loop is like calculating the partition function with additional imaginary charged instanton. Hence in the same way as the partition function, we can rewrite this in terms of a contour integral. In the l -instanton sector (i.e. number of non-zero m's is l) , we devide the place where the imaginary chage put on . One is k with nonzero m_k , another is k with zero m_k . Hence the above equation becomes

$$\begin{aligned}
&= e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} e^{-\frac{n^2 A_1 A_2}{2AN}} \frac{1}{N} \sum_{l=1}^N {}_N C_l \sum_{m_1, \dots, m_l \neq 0} e^{m_1 + \dots + m_l} e^{-\frac{2\pi^2 N}{A} (m_1^2 + \dots + m_l^2)} \\
&\quad \times [l e^{-2\pi i n m_1 \frac{A_2}{A}} \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\
&\quad \quad \times \Delta(y_1 + 2\pi m_1 + inA_2/N, y_2 + 2\pi m_2, \dots, y_l + 2\pi m_l, y_{l+1}, \dots, y_N) \\
&\quad \quad \times \Delta(y_1 - 2\pi m_1 + inA_1/N, y_2 - 2\pi m_2, \dots, y_l - 2\pi m_l, y_{l+1}, \dots, y_N) \\
&\quad + (N-l) \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\
&\quad \quad \times \Delta(y_1 + 2\pi m_1, \dots, y_l + 2\pi m_l, y_{l+1} + inA_2/N, y_{l+2}, \dots, y_N) \\
&\quad \quad \times \Delta(y_1 - 2\pi m_1, \dots, y_l - 2\pi m_l, y_{l+1} + inA_1/N, y_{l+2}, \dots, y_N)]. \quad (27)
\end{aligned}$$

The first integral in the big parentheses, using the preceeding method, has the following contour integral representation :

$$\begin{aligned}
& \frac{(N-l)!}{N!} Z_G \left(\frac{N}{A}\right) \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_l}{2\pi i} e^{-\frac{4\pi^2 N}{A} (m_1^2 t_1 + \dots + m_l^2 t_l)} \left(1 + \frac{1}{t_1}\right)^N \dots \left(1 + \frac{1}{t_l}\right)^N \\
& \quad \times e^{2\pi i n m_1 \frac{A_1 - A_2}{A} t_1 - \frac{n^2}{N} \frac{A_1 A_2}{A} t_1} \det^{(l)} B. \quad (28)
\end{aligned}$$

The second integral also has the contour integral representation as

$$\begin{aligned} & \frac{(N-l-1)!}{N!} Z_G \left(\frac{N}{A} \right) \oint \frac{dt_1}{2\pi i} \cdots \oint \frac{dt_{l+1}}{2\pi i} e^{-\frac{4\pi^2 N}{A} (m_1^2 t_1 + \cdots + m_{l+1}^2 t_{l+1})} (1 + \frac{1}{t_1})^N \cdots (1 + \frac{1}{t_{l+1}})^N \\ & \times e^{-\frac{n^2}{N} \frac{A_1 A_2}{A} t_{l+1}} (1 + \frac{1}{t_{l+1}})^N \det^{(l+1)} C, \end{aligned} \quad (29)$$

where

$$B_{ij} = \frac{1}{\frac{b_j}{b_i}(1+t_j) - t_i} , \quad b_i = -2\pi m_i + \frac{inA_1}{N} \delta_{i1}, \quad (30)$$

$$C_{ij} = \frac{1}{\frac{c_j}{c_i}(1+t_j) - t_i} , \quad c_i = -2\pi m_i (1 - \delta_{i,l+1}) + \frac{inA_1}{N} \delta_{i,l+1}. \quad (31)$$

For the sake of numerical calculation, we further rewrite it to a exponential form. The first term of the l -instanton contribution to the Wilson loop has the following form:

$$\sum_{m_1, \dots, m_l \neq 0} \epsilon^{m_1 + \cdots + m_l} \oint \prod_{i=1}^l \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} f(m_1, t_1) \det^{(l)} B. \quad (32)$$

In the determinant we assume the cycle including the index 1 to have length k and explicitly to be $(1, a_1, \dots, a_{k-1})$. Thus the determinant will be

$$\det^{(l)} B = \sum_{k=1}^l \sum_{a_1 \neq \cdots \neq a_{k-1} \neq 1} B_{1a_1} \cdots B_{a_{k-1}1} \det^{(l-k)} B, \quad (33)$$

here $\det^{(l-k)} B$ is the determinant not including the index $1, a_1, \dots, a_{k-1}$. Under the index rename invariant measure, we can rename a_1, \dots, a_{k-1} to be $2, \dots, k$. Thus we get

$$\det^{(l)} B \rightarrow \sum_{k=1}^l \frac{(l-1)!}{(l-k)!} (-1)^{k-1} B_{12} B_{23} \cdots B_{k1} \det^{(l-k)} B \quad (34)$$

Hence eq.(32) becomes

$$\begin{aligned} & \sum_{m_1, \dots, m_l \neq 0} \epsilon^{m_1 + \cdots + m_l} \oint \prod_{i=1}^l \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} f(m_1, t_1) \det^{(l)} B \\ & = l! \sum_{k=0}^{l-1} \frac{1}{k!} \beta_{l-k}(A_1, A_2) \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \cdots + m_k} \oint \prod_{i=1}^k \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} \det^{(k)} M. \end{aligned} \quad (35)$$

where

$$\begin{aligned} \beta_k(A_1, A_2) & \equiv (-1)^{k-1} \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \cdots + m_k} \oint \prod_{i=1}^k \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} B_{12} B_{23} \cdots B_{k1} \\ & \times e^{-2\pi i n m_1 \frac{A_2}{A} + 2\pi i n m_1 \frac{A_1 - A_2}{A} t_1 - \frac{n^2}{N} \frac{A_1 A_2}{A} t_1} \end{aligned} \quad (36)$$

is the type 1 connected amplitude of k -instanton

In the same way , under the index rename invariant measure ,we get

$$\det^{(l+1)} C \rightarrow \sum_{k=1}^{l+1} \frac{l!}{(l-k+1)!} (-1)^{k-1} C_{12} C_{23} \cdots C_{l+11} \det^{(l+1-k)} C. \quad (37)$$

Hence the second term of the l -instanton contribution to the Wilson loop becomes

$$\begin{aligned} & \sum_{m_1, \dots, m_l \neq 0} \epsilon^{m_1 + \dots + m_l} \oint \prod_{i=1}^l \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} \oint \frac{dt_{l+1}}{2\pi i} e^{-\frac{n^2 A_1 A_2}{NA} t_{l+1}} (1 + \frac{1}{t_{l+1}})^N \det^{(l+1)} C \\ &= l! \sum_{k=0}^l \frac{1}{k!} \gamma_{l-k}(A_1, A_2) \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \dots + m_k} \oint \prod_{i=1}^k \frac{dt_i}{2\pi i} e^{-N\tilde{\Phi}_{m_i}(t_i)} \det^{(k)} M. \end{aligned} \quad (38)$$

where

$$\begin{aligned} \gamma_k(A_1, A_2) &\equiv (-1)^k \sum_{m_1, \dots, m_k \neq 0} \epsilon^{m_1 + \dots + m_k} \oint \prod_{i=1}^{k+1} \frac{dt_i}{2\pi i} \prod_{i=1}^k e^{-N\tilde{\Phi}_{m_i}(t_i)} \\ &\quad \times e^{-\frac{n^2 A_1 A_2}{NA} t_{k+1}} (1 + \frac{1}{t_{k+1}})^N C_{12} C_{23} \cdots C_{k+11}. \end{aligned} \quad (39)$$

is the type 2 connected amplitude of k -instanton. Using the fact that k -instanton amplitude can be rewritten as eq (21), we finaly obtain the following result:

$$\begin{aligned} & W_n(A_1, A_2) Z(A) / Z_{\text{weak}}(A) e^{-\frac{n^2 A_1 A_2}{2AN}} \\ &= \frac{1}{N} \sum_{l=0}^N \oint \frac{dz}{2\pi i} \left(\sum_{k=0}^{l-1} \frac{\beta_{l-k}(A_1, A_2)}{z^{k+1}} + \sum_{k=0}^l \frac{\gamma_{l-k}(A_1, A_2)}{z^{k+1}} \right) \exp \left(\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j(A) \right) \end{aligned} \quad (40)$$

This is the base of our numerical calculation of the Wilson loop average.

For example , in the 0-instanton sector only γ_0 contributes to the Wilson loop. We get

$$W_n(A_1, A_2) Z(A) / Z_{\text{weak}}(A) e^{-\frac{n^2 A_1 A_2}{2AN}}|_{0-\text{inst}} = \frac{1}{N} \gamma_0 = \frac{1}{N} \oint \frac{dt}{2\pi i} e^{-\frac{n^2 A_1 A_2}{NA} t} (1 + \frac{1}{t})^N. \quad (41)$$

By rescaling $t \rightarrow Nt$, we obtain the known result of the Wilson loop in the weak coupling phase at the large N [10] :

$$W_n(A_1, A_2)|_{0-\text{inst}} = \frac{1}{n} \sqrt{\frac{A}{A_1 A_2}} J_1(2n \sqrt{\frac{A_1 A_2}{A}}) + O(1/N). \quad (42)$$

So the multi-instanton effect is essential for the Wilson loop in the strong coupling phase. Apparently if we extrapolate the above equation to $A \rightarrow \infty$ ³ with fixed A_1 , we cannot obtain the simple area law on infinite disk but obtain a oscilating behavior.

³The phase transition occurs before $A \rightarrow \infty$.

4 Effects of 1-instanton

In this section we consider the contribution of one instanton to the partition function[6] and the Wilson loop. We see that the 1-instanton effect is insufficient to explain the large N phase transition. We use the word “1-instanton” to mean that only $\alpha_1, \beta_1, \gamma_0, \gamma_1$ are included in the partition function and the Wilson loop. So this is a kind of dilute gas approximation in our system.

The dilute gas approximation leads to the following result for the partition function.

$$\frac{Z(A)}{Z_{\text{weak}}(A)} \rightarrow \oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1-z} \exp(z\alpha_1(A)) = \sum_{k=1}^N \frac{(\alpha_1(A))^k}{k!}. \quad (43)$$

We should note since maximum number of the instanton is N , the partition function in this approximation is not completely exponentiated as $\exp(\alpha_1)$. In the large N limit, we can apply the saddle point method to α_1 . The saddle point equation $\Phi'_m(t) = 0$ has two solutions

$$t = t_{\pm} \equiv \frac{-1 \pm \sqrt{1 - \frac{A}{\pi^2 m^2}}}{2}. \quad (44)$$

We see quite different behavior according to $A > \pi^2 m^2$.

For the case of $A < \pi^2 m^2$, both saddle points exist on negative real axis and satisfy

$$\Phi_m(t_-) < \Phi_m(t_+). \quad (45)$$

Nevertheless only t_+ saddle point is selected since t_- steepest decent line is not consistent with the original contour. Hence we obtain in the large N limit ,

$$\begin{aligned} \oint \frac{dt}{2\pi i} e^{-N\tilde{\Phi}_m(t)} &\simeq (-) \frac{1}{\sqrt{2\pi N |\Phi_m^{(2)}(t_+)|}} e^{-N\tilde{\Phi}_m(t_+)} \\ &= \frac{(-)^{N-1}}{\sqrt{2\pi N \frac{16\pi^4 m^4}{A^2} \sqrt{1 - \frac{A}{\pi^2 m^2}}}} e^{-\frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})}. \end{aligned} \quad (46)$$

where $\gamma(x)$ is defined as [6]

$$\gamma(x) \equiv \sqrt{1-x} - \frac{x}{2} \log \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}, \quad (47)$$

Since $\gamma(x)$ is positive real for $x < 1$, in this region 1-instanton amplitude is strongly suppressed with order $\frac{1}{\sqrt{N}} e^{-N}$.

For the case of $A > \pi^2 m^2$, both saddle points must be selected because t_{\pm} steepest descent lines are consistent with original contour and satisfy

$$\text{Re}[\Phi_m(t_+)] = \text{Re}[\Phi_m(t_-)]. \quad (48)$$

Therefore, we get

$$\oint \frac{dt}{2\pi i} e^{-N\tilde{\Phi}_m(t)} \simeq \frac{(-)^N}{\sqrt{2\pi N \frac{16\pi^4 m^4}{A^2} \sqrt{\frac{A}{\pi^2 m^2}} - 1}} (e^{i\frac{3}{4}\pi - \frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})} + e^{-i\frac{3}{4}\pi + \frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})}). \quad (49)$$

Since $\gamma(x)$ are pure imaginary in $x > 1$, the above equation has order $N^{-1/2}$ in the large N limit.

From the above observation the following truncation about the instanton charge is justified. If A satisfies the relation $k^2\pi^2 < A < (k+1)^2\pi^2$ (k : positive integer), we truncate the instanton charge as $-k \leq m \leq k$. That is, According to A , the instanton charges which are included in the large N limit are changed stepwise as figure1.

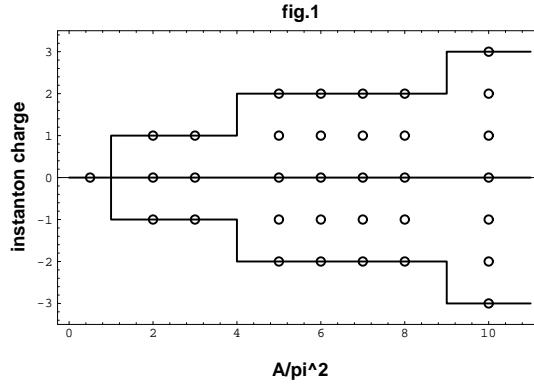


Figure1: The instanton charges which must be included in the large N limit (indicated by circle) are changed stepwise according to A .

This truncation is basically justified in the multi-instanton amplitude because α_k become factorized as $(\alpha_1)^k$ in the interaction free approximation. But we will see in the next section a peculiar dynamics in the multi-instanton interaction make further truncation justified in the large N limit.

Since α_1 has order $N^{-1/2}$ in $A \geq \pi^2$, we may replace eq (43) to $\exp(\alpha_1)$. But this gives order $N^{-5/2}$ result for the free energy $\tilde{F} \equiv \frac{1}{N^2} \log[Z/Z_{\text{weak}}]$. To see explicitly that

the dilute gas approximation is insufficient we show in figure 2 the $N = 5$ result of the free energy with the dilute gas approximation and the large N Douglas-Kazakov solution which is described later .

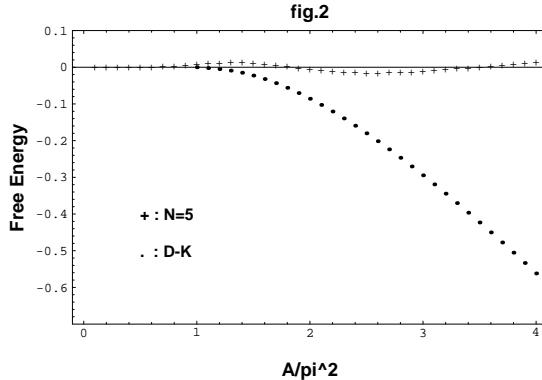


Figure2: The free energy of $N = 5$ in the dilute gas approximation and the large N D-K solution.

Next we consider the dilute gas approximation of the Wilson loop. This lead to

$$W_n(A_1, A_2) \rightarrow \left[\frac{1}{N} \gamma_0(A_1, A_2) + \frac{1}{N} (\beta_1(A_1, A_2) + \gamma_1(A_1, A_2)) \frac{\sum_{k=1}^{N-1} \frac{(\alpha_1(A))^k}{k!}}{\sum_{k=1}^N \frac{(\alpha_1(A))^k}{k!}} \right] e^{-\frac{n^2 A_1 A_2}{2AN}} \quad (50)$$

for the Wilson loop. Thus we consider β_1 and γ_1 . In the same way as α_1 ,the large N saddle point method leads to $\beta_1, \gamma_1 = O(\frac{1}{\sqrt{N}} e^{-N})$ for $A < \pi^2$ and $\beta_1, \gamma_1 = O(\frac{1}{\sqrt{N}})$ for $A > \pi^2$. Here we use the rescaling $t_2 \rightarrow Nt_2$ for γ_1 . Thus in the large N limit the dilute gas approximation lead to the value in the weak coupling phase. To see explicitly the insufficiency of the dilute gas approximation we show in figure 3 the $N = 5$ results of the Wilson loop average

$$\widetilde{W}_1(A_1, A_2) \equiv W_1(A_1, A_2) - \frac{1}{N} \gamma_0(A_1, A_2) e^{-\frac{A_1 A_2}{2AN}} \quad (51)$$

in the approximation and compare the results with the large N Boulatov,Daul-Kazakov (BDK) solution

$$\widetilde{W}_1^{\text{BDK}}(A_1, A_2) \equiv W_1^{\text{BDK}}(A_1, A_2) - \sqrt{\frac{A}{A_1 A_2}} J_1\left(2\sqrt{\frac{A_1 A_2}{A}}\right). \quad (52)$$

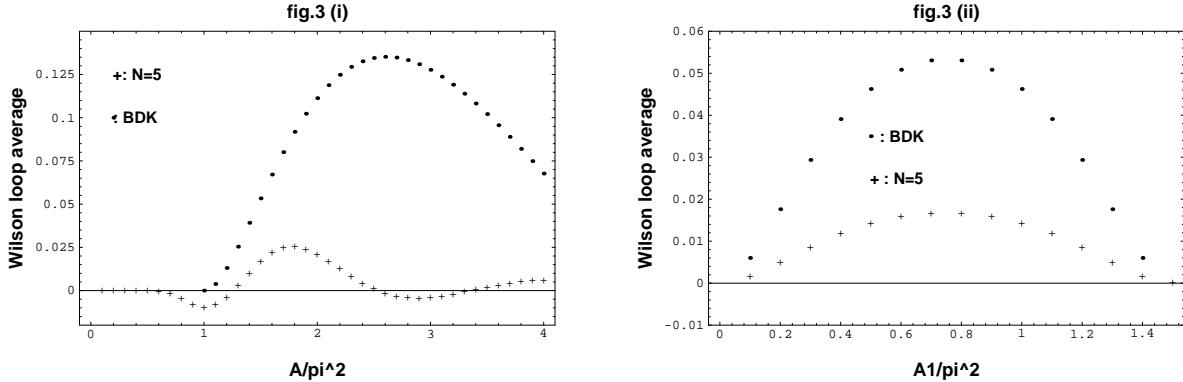


Figure3: The Wilson loop average of $N = 5$ in the dilute gas approximation and the large N BDK solution with (i) $A_1 = A/2$ (ii) $A = 1.5\pi^2$

5 Large N neutrality

In the preceeding sections we gave the contour integral representation for the multi-instanton sector of the partition function and the Wilson loop. The representation has a simple gauge group parameter (N) dependence. That is , α, β, γ have the form such as :

$$\oint \prod dt e^{-Nf(t)} g(t). \quad (53)$$

where $g(t)$ depends not on N for α_k and weakly depends on N for β_k, γ_k . Then ,If we consider the large N limit, the integrals for α, β, γ are dominated by the saddle points given by the solution of $\partial f(t) = 0$. Fortunately, it is no need for solving the complicated saddle point equations corresponded to the multi-dimensional contour integral in our case. The saddle point equations are decoupled for each variables and the solutions of the equations are sets of each one dimensional solution which was found by Gross and Matytsin [6]. Then if we neglect the interaction term $M_{12}M_{23} \cdots M_{j1}$ (for example α_j 's case), α_j becomes factorized such as $(\alpha_1)^j$ and the behavior change from exponential damping to oscilating at $A = \pi^2 m^2$ observed in the preceeding section still hold for the multi-instanton amplitude. But there is a remarkable feature in the interaction term of the multi-instanton sector.

We point out that in the region of $A > \pi^2$ special charge configurations of the multi-instanton exist. For this purpose let us first consider the following toy contour

integral:

$$\oint \frac{dt}{2\pi i} e^{-N\phi(t)} \frac{1}{(t-t_c)^n}, \quad (54)$$

where t_c is the saddle point of $\phi(t)$. We can approximate the above equation in the large N limit to a principal value integral along the steepest descent line and a contour integral along small semi circle around the saddle point. As the result we obtain finite value with order $N^{\frac{n-1}{2}} \exp(-N\phi(t_c))$. We call this type of problem “singular saddle point problem”. Apparently, the degree of singularity in $\frac{1}{(t-t_c)^n}$ at the saddle point determines the power of N .

In our case such problem occurs in a complicated manner. Let's consider the singularity of α_k . The maximal singularity occurs when t 's are the solution of $M_{ii+1}^{-1} = 0$. That is

$$\begin{pmatrix} -1 & \frac{m_2}{m_1} & 0 & \cdots & 0 \\ 0 & -1 & \frac{m_3}{m_2} & 0 & 0 \\ \vdots & & & \vdots & \\ \frac{m_1}{m_k} & 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} -\frac{m_2}{m_1} \\ -\frac{m_3}{m_2} \\ \vdots \\ -\frac{m_1}{m_k} \end{pmatrix}. \quad (55)$$

The consistency of the above equation requires $m_1 + \cdots + m_k = 0$ i.e. neutral configuration. Under this condition the equation has a one parameter solution given by

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{k-1} \\ t_k \end{pmatrix} = \begin{pmatrix} \frac{m_k}{m_1}(t+1) + \frac{m_{k-1} + \cdots + m_2}{m_1} \\ \frac{m_k}{m_2}(t+1) + \frac{m_{k-1} + \cdots + m_3}{m_2} \\ \vdots \\ \frac{m_k}{m_{k-1}}(t+1) \\ t \end{pmatrix}. \quad (56)$$

For the case of even k if we consider the following configurations

$$(m_1, m_2, m_3, m_4, \dots, m_k) = (m, -m, m, -m, \dots, -m) \quad (57)$$

which we call “neutral”, the above solution simply becomes

$$t_{\text{even}} = t, \quad t_{\text{odd}} = -(t+1). \quad (58)$$

On the other hand ,under the above charge configuration , the large N saddle points for any t_i are same and given by :

$$t_{\pm} = \frac{-1 \pm \sqrt{1 - \frac{A}{\pi^2 m^2}}}{2}, \quad t_+ + t_- + 1 = 0. \quad (59)$$

When $A > m^2\pi^2$, both saddle points must be selected for each variable. Hence the saddle points include the following combinations:

$$t_{\text{even}} = t_{\pm} \quad , \quad t_{\text{odd}} = t_{\mp} = -(t_{\pm} + 1). \quad (60)$$

In this case the above saddle points ride on the line of maximal singularity and thus we encounter the singular saddle point problem. We can also show the “neutral” configuration is the only case in which the saddle point completely rides on the line of the maximal singularity. Hence the neutral configuration are dominant in the large N limit of α_k . Compared to the one dimensional toy model, we have not the point like singularity but the line like singularity and the steepest descent line is also singular in this case. This makes a trouble in estimating the power of N in α_k . But we show in the next section numerical calculation indicate that for even k α_k has linear scaling with respect to N .

For the case of odd k , the saddle points can not completely ride on the line of maximal singularity. But if we weaken the singularity, there is a set of configurations:

$$\begin{aligned} & (m_1, m_2, m_3, m_4, \dots, m_{k-1}, m_k) \\ &= \{(m, m, -m, m, \dots, m, -m), (m, -m, -m, m, \dots, m, -m), \\ & \quad (m, -m, m, m, \dots, m, -m), \dots, (m, -m, m, -m, \dots, m, m)\} \end{aligned} \quad (61)$$

which we call “next neutral”. Under these configurations $M_{ii+1}^{-1} = 0$ ($i = 1, \dots, \check{j}, \dots, k$) are satisfied by the following combinations of saddle points in the region of $A > m^2\pi^2$

$$(t_1, \dots, \check{t}_j, \dots, t_k) = (t_{\pm}, t_{\mp}, \dots, t_{\mp}). \quad (62)$$

Apparently the number of the “next neutral” configurations is k for fixed m . These configurations are dominant in the large N limit of α_k . The numerical calculation in the next section shows that α_{odd} seems to oscillate with $N^{-1/2}$ scaling.

Next we consider β_k, γ_k i.e. the connected k -instanton contribution to the Wilson loop. Because we now consider large N limit and the fact that the additive imaginary charge in our representation of the Wilson loop has order $1/N$, the singular saddle point problem occurs in the same manner as α_k . Thus we conclude that the dominant instanton configurations are “neutral” for $\beta_{\text{even}}, \gamma_{\text{even}}$ and “next neutral” for $\beta_{\text{odd}}, \gamma_{\text{odd}}$.

The numerical calculation in the next section show that $\beta_{\text{even}}, \gamma_{\text{even}}$ have also linear scaling (N) and $\beta_{\text{odd}}, \gamma_{\text{odd}}$ have the oscilated $N^{-1/2}$ scaling.

From the above argument , we find a peculiar phenomena about the multi-instanton configuration in the large N limit. We call the phenomena “large N neutrality”. By “neutrality”, we would not intend general neutral configuration which satisfy $\sum_{i=1}^N m_i = 0$, but intend that all instantons have equal absolute charge and number of positive charge instantons and number of negative charge instantons are same or at most differs by one.

6 Numerical calculation

In this section we show the results of numerical calculation for the connected multi-instanton amplitudes (α' s, β' s, γ' s) , the free energy and the Wilson loop for finite N . We assume $0 < A < 4\pi^2$. From the above argument, in this region we can restrict the instanton charge as $m_i = \pm 1$ in the multi-instanton sector. In the numerical calculation we dont perform the integral numerically because we must deal with highly fluctuated integral like the Fourier transformation. But we perform the residue summation which is finite series for finite N in our case.

First, we consider α_k . Fig1 shows the numerical results of $\alpha_i(A)(i = 1, \dots, 4)$ with various N and A . We used the following approximation . For α_{even} we restrict the “neutral” configuration eq.(56) with $m = \pm 1$ and for α_{odd} restrict the “next neutral” configurations eq.(60) with $m = \pm 1$ which are dominant in the large N limit.

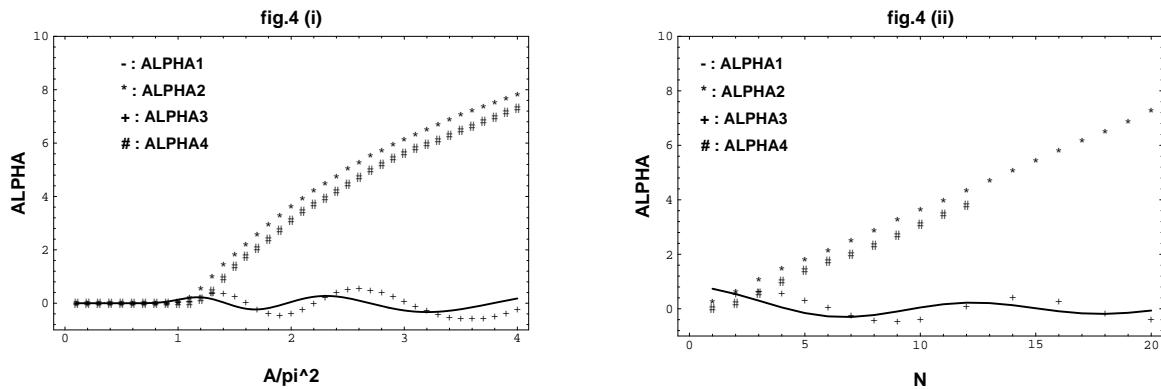


Figure4: The connected multi-instanton amplitude α_j up to $j = 4$ with (i) $N = 10$ and

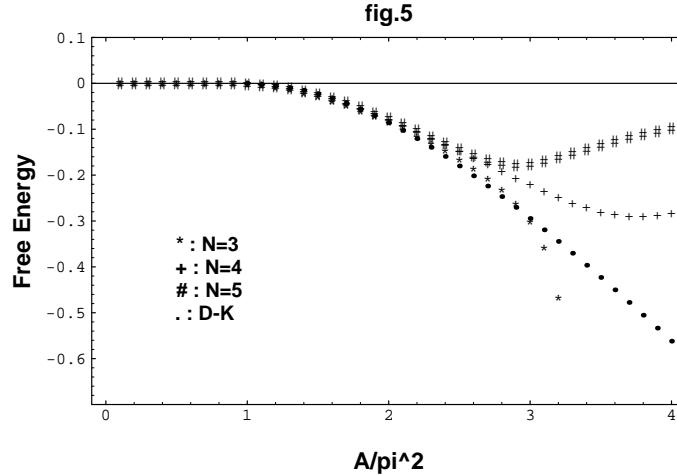
various A/π^2 and (ii) $A/\pi^2 = 2$ and various N . There are remarkable difference between α_{even} and α_{odd} .

We observe that compared to the α 's in $A > \pi^2$, the α 's in $A < \pi^2$ are negligible. Moreover we see remarkable difference between α_{even} and α_{odd} with respect to order and shape in the region of $A > \pi^2$. The shapes of α_2, α_4 are similar and both have the properties of non-oscillating and linear scaling with respect to N . On the other hand α_1, α_3 have the properties of oscillating and $N^{-1/2}$ scaling. We already observed that α_1 has $N^{-1/2}$ scaling and α_2 has linear(N) scaling using the large N saddle point method [9]. Hence from the argument in the previous section and these numerical results, we conjecture that α_{even} have linear(N) scaling but α_{odd} have $N^{-1/2}$ scaling in the large N limit. So there is “democracy” in the interaction between the instantons. That is, α_2 which is the 2-body force, α_4 which is the 4-body force, ... are same order with respect to N .

Figure 5 shows the free energy by our “neutral” multi-instanton scheme

$$\tilde{F}(A) = \frac{1}{N^2} \log \left[\frac{Z(A)}{Z_{\text{weak}}(A)} \right] \rightarrow \frac{1}{N^2} \log \left[\oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1-z} \exp \left(- \sum_{n=1}^{[N/2]} \frac{1}{2n} z^{2n} \alpha_{2n}(A) \right) \right] \quad (63)$$

for $N=3, 4$ and 5 and the large N exact solution of the free energy obtained by Douglas-Kazakov. Here we neglect α_{odd} from the above observation.



	$A = 1.2\pi^2$	$A = 1.4\pi^2$	$A = 1.6\pi^2$	$A = 1.8\pi^2$	$A = 2\pi^2$
N=3	0.009489	-0.02186	-0.03949	-0.06146	-0.08697
N=4	-0.006966	-0.01797	-0.03448	-0.05545	-0.07973
N=5	-0.005588	-0.01580	-0.03173	-0.05231	-0.07632
DK	-0.005064	-0.01497	-0.03213	-0.05609	-0.08588

Figure5: The free energy for $N = 3, 4, 5$ approximated by the “neutral” configurations with $m = \pm 1$ and the large N exact solution by Douglas-Kazakov. The neutral multi-instanton picture well describe the behavior of the large N D-K solution in the neighborhood of the phase transition point.

The D-K solution is given by the following implicit equations:

$$F'(A) = -\frac{a^2}{2} + \frac{1}{12}a^2\left(1 - \frac{b^2}{a^2}\right) + \frac{1}{24} + \frac{1}{96}a^4\left(1 - \frac{b^2}{a^2}\right)^2 A, \quad (64)$$

$$A = 4K\left(\frac{b}{a}\right)\left(2E\left(\frac{b}{a}\right) - \left(1 - \frac{b^2}{a^2}\right)K\left(\frac{b}{a}\right)\right), \quad (65)$$

$$a = 4K\left(\frac{b}{a}\right)/A \quad (66)$$

where $F(A) \equiv \frac{1}{N^2} \log Z(A)$ with the boundary condition $F(\pi^2) = \frac{1}{N^2} \log Z_{\text{weak}}(\pi^2)$ and $K(x), E(x)$ are the complete elliptic integral of first, second kind respectively. The above equations are obtained BIPZ-like analysis [14] of the heat kernel representation of the partition function.

The partiton function for $N=3$ in our approximation turns into unphysical region in $A \gtrsim 3.2\pi^2$ and thus abort in this region. We observe that the graphs for finite N stand up at $A \simeq \pi^2$ and in the neighborhood of $A = \pi^2$ the results for finite N seem to converge on the large N D-K solution as N becomes large. But in the region $A \gtrsim 2\pi^2$ discrepancy of the finite N graphs and the D-K solution becomes large. Since we can see that the further approximation with respect to absolute value of instanton charge does not make the discrepancy small, we conjecture that the instanton summation form of the partition function is a kind of asymptotic series in the region of $A > \pi^2$.

Next we consider the β_k, γ_k . We follow the same approximation scheme as the α_k . That is, for $\beta_{\text{even}}, \gamma_{\text{even}}$ we restrict the “neutral” configurations with $m = \pm 1$ and for $\beta_{\text{odd}}, \gamma_{\text{odd}}$ restrict the “next neutral” configurations with $m = \pm 1$.

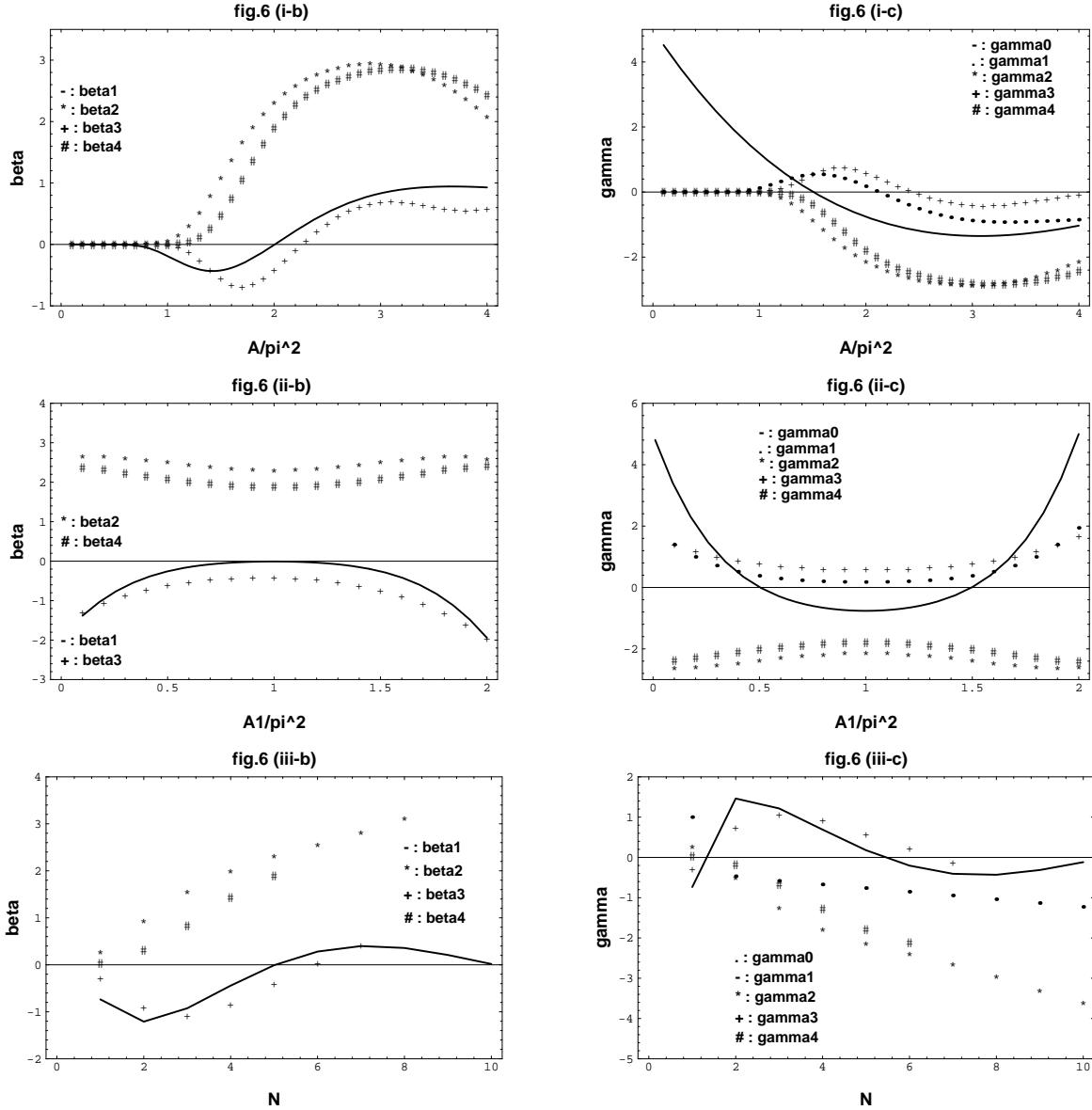


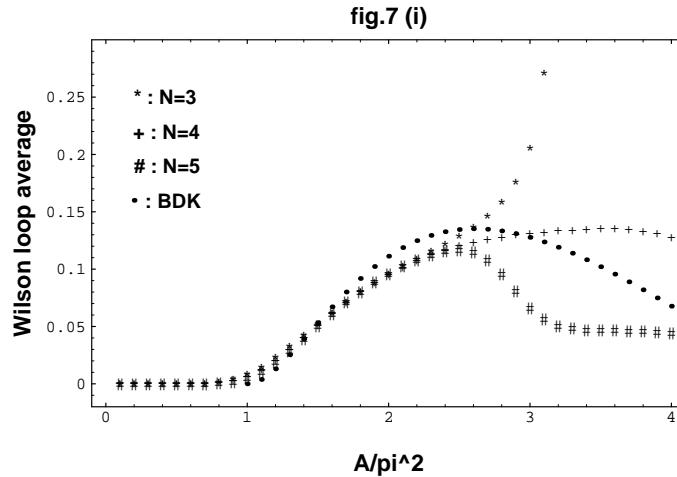
Figure6: The contribution of the connected multi-instanton amplitude β_j, γ_j to the Wilson loop with (i) $N = 5$, various $A = 2A_1$ (ii) $N = 5, A = 2\pi^2$ various A_1 (iii) $A = 2\pi^2, A_1 = \pi^2$, various N are shown

Figure6 show $\beta_j (j = 1, \dots, 4)$ and $\gamma_j (j = 0, \dots, 4)$ with various N, A, A_1 . In fig.6 we observe that compared to the β_j, γ_j in $A > \pi^2$, the β_j, γ_j in $A < \pi^2$ are also negligible. But compared to α_j there is no remarkable difference with respect to order between β_{even} and β_{odd} and between γ_{even} and γ_{odd} in figure6(i)(ii). We should note these are $N = 5$ results. And we show in fig.6(iii) both $\beta_{\text{even}}, \gamma_{\text{even}}$ have linear scaling with respect to N contrasted to oscillated damping behavior of $\beta_{\text{odd}}, \gamma_{\text{odd}}$. Thus in the large N limit only $\beta_{\text{even}}, \gamma_{\text{even}}$ are survived and the approximation in which $\beta_{\text{odd}}, \gamma_{\text{odd}}$ are neglected for finite N is more nearer to the large N exact solution than the approximation in which all β, γ are included.

Figure7 show the simple Wilson loop average (i.e. winding $n = 1$) for $N = 3, 4, 5$ by our “neutral” multi-instanton scheme :

$$\begin{aligned} & \widetilde{W}_1(A_1, A_2) \\ &= e^{-\frac{A_1 A_2}{2AN}} \left[\frac{\frac{1}{N} \sum_{l=0}^N \oint \frac{dz}{2\pi i} \left(\sum_{k=0}^{l-1} \frac{\beta_{l-k}(A_1, A_2)}{z^{k+1}} + \sum_{k=0}^{l-1} \frac{\gamma_{l-k}(A_1, A_2)}{z^{k+1}} \right) \exp\left(-\sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2n} z^{2n} \alpha_{2n}(A)\right)}{\oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}(1-z)} \exp\left(-\sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2n} z^{2n} \alpha_{2n}(A)\right)} \right] \end{aligned} \quad (67)$$

where $\beta_{\text{odd}}, \gamma_{\text{odd}}$ are neglected and the large N exact solution obtained by Boulatov and Daul-Kazakov (BDK)[10].



	$A = 1.2\pi^2$	$A = 1.4\pi^2$	$A = 1.6\pi^2$	$A = 1.8\pi^2$	$A = 2\pi^2$
N=3	0.02343	0.04265	0.06227	0.08011	0.09564
N=4	0.02049	0.04051	0.06132	0.07992	0.09525
N=5	0.01861	0.03919	0.06085	0.08014	0.09589
BDK	0.01307	0.03918	0.06712	0.09186	0.11130

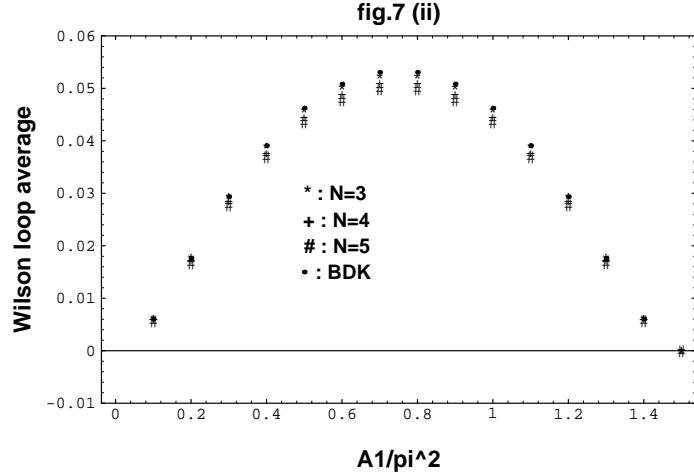


Figure7: The Wilson loop average with (i) $N = 3, 4, 5$, various $A = 2A_1$ (ii) $N = 3, 4, 5, A = 1.5\pi^2$ various A_1 by our “neutral” multi-instanton scheme and for each case showed the large N Boulatov,Daul-Kazakov solution. The neutral multi-instanton scheme well describes the behavior of the large N BDK solution in the neighborhood of the phase transition point

The large N BDK solution is also given by the following implicit equation :

$$W_1(A_1, A_2) = \oint_C \frac{dz}{2\pi i} e^{A_1 z - f(z)}. \quad (68)$$

where $f(z)$ is the resolvent in this system and the contour C encircles the cut of $f(z)$ counter-clockwise. From the above equation the difference of the phase in the Wilson loop average is concentrated on the resolvent . The resolvent is given by the famous Wigner’s semi-circle law in the weak coupling phase ($A < \pi^2$):

$$f(z) \equiv \frac{Az}{2} - \frac{A}{2} \sqrt{z^2 - 4/A} \quad (69)$$

On the other hand in the strong coupling phase the resolvent is given by

$$f(z) \equiv \frac{Az}{2} + \frac{2}{az} \sqrt{(a^2 - z^2)(b^2 - z^2)} \pi\left(-\frac{b^2}{z^2}, \frac{b}{a}\right) \quad (70)$$

where a, b are the same parameter as the equations of the free energy and $\pi(c, x)$ is the complete elliptic integral of third kind.

We observe that the graphs for finite N stand up at $A \simeq \pi^2$ and in the neighborhood of $A = \pi^2$ the results for finite N seem to converge on the large N BDK solution as N becomes large. But we also observe that the disagreement becomes large in the region $A > 2\pi^2$ which is same as the free energy. Hence the instanton summation form of the Wilson loop is also a kind of asymptotic series and divergent in the region of $A > \pi^2$.

Hence we conclude that viewed from weak coupling phase the large N phase transition is physically caused by the “neutral” multi-instanton.

7 conclusion

We have analyzed the multi-instanton amplitudes in terms of the contour integral representation and found that the representation make clear the properties and the role of the multi-instanton in the large N phase transition. In particular the “neutral” configurations of the even number instantons are essential. We also showed that our “neutral” multi-instanton scheme well describes the behavior of the free energy and the Wilson loop around the critical point $A_c = \pi^2$ up to $A \simeq 2\pi^2$. And we show that is impossible in the dilute gas approximation.

But we observe the agreement region of our results and the large N exact solution tends to become narrow as N becomes large. In addition we can see that the further approximation with respect to absolute value of instanton charge does not make the discrepancy small. So we conjecture the form of the instanton summation is a kind of asymptotic series. For example ,assume that $S(z)$ is defined nonperturbatively and has an asymptotic expansion $S(z) \Rightarrow \sum_{k=0}^{\infty} s_k z^k$. By truncating the series with appropriate region of summation in the neighborhood of $z = 0$, we can approach to the true value. The same scenario works for our case of the “neutral” multi-instanton truncation. Thus we conclude that our “neutral” multi-instanton scheme is a good truncation in the strong coupling region up to $A \simeq 2\pi^2$ at large N .

Recently the field strength correlators in the large N limit are determined [15] using the abelianization technique for the path integral[16]. To explore whether our method can describe the behavior of these correlators is our next task.

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